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ON THE STRUCTURES OF Σ-SEMIGROUP

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Abstract

In this paper, we provide a new structure of a locally associative algebraic structures that posses the properties of both left almost semigroup and right almost semigroup. A homomorphism function is defined on the new structure which helped to establish that every $\sum$*-Semigroup is left nuclear or right nuclear square. The results further showed that the new structures formed is transitively commutative. The work also defined another function on the structure which helped to obtain a directed graph in which the rows of the adjacency matrix formed are linearly dependent vectors.*

Keywords: *Left almost semigroup, right almost semi group, commutative groups and nuclear square.*

Introduction

A semigroup is an algebraic structure consisting of a set (groupoid) together with an associative binary operation. The binary operation of a semigroup is most often denoted multiplicatively by $x \cdot y$, or simply $x\ y$, and associativity is formally expressed as $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ for all x, y and z in the semigroup. A left almost semigroup abbreviated as LA semigroup is an algebraic structure midway between a groupoid and a commutative semigroup. In accordance with [18] and [1], the structure was introduced by [10]. The structure was named Abel Grassmann's groupoid abbreviated as AG groupoid [14]. The structure is infact the generalization of commutative semigroups [9].

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DEFINITION OF SOME BASIC NOTATIONS USED

Definition 1.6.1 A groupoid is an algebraic structure (G, ∗) consisting of a non-empty set G and a binary partial function ' ∗ ' defined on G.

A groupoid *G* is called left almost semigroup if it satisfies the left invertive law, that is for all a, b, c in G (a.b).c = (c.b).a. It is also called Abel-Grassmann's groupoid, abbreviated as AG-Groupoid.

Definition 1.6.2 Left translation: let G be an LA semigroup and a in G, then a mapping $L_a : G \rightarrow G$ defined by $L_a(x) = ax$ is called left translation by element a.

Definition 1.6.3 Right Translation: let G be an LA semigroup and a in G, then a mapping $R_a : G \rightarrow G$ defined by $R_a(x) = xa$ is called right translation by a.

Definition 1.6.4 Left Cancellative LA semigroup: an LA semigroup G is called left Cancellative if all the left translations are injective.

Definition 1.6.5 Right Cancellative LA semigroup: An LA semigroup is called Right cancellative if all the right translations are injective.

Definition 1.6.6 Cancellative LA semigroup: an LA semigroup is called cancellative if all the translations are injective.

Definition 1.6.6.1 Every group is a cancellative semigroup.

Definition 1.6.6.2 The set of positive integers under addition is a cancellative semigroup.

Definition 1.6.6.3 The set of nonnegative integers under addition is a cancellative monoid.

Definition 1.6.6.4 The set of positive integers under multiplication is a cancellative monoid.

Definition 1.6.6.5 Let '*S*' be the semigroup of real square matrices of order *n* under matrix multiplication. Let '*a*' be any element in *S*. If *A* is nonsingular then *A* is both left cancellative and right cancellative. If '*A*' is singular then *A* is neither left cancellative nor right cancellative.

Definition 1.6.7 A Groupoid which is both left almost and right almost semigroup is called an almost semigroup.

Definition 1.6.8 A semigroup is non empty set G together with an associative binary operation * such that $\forall a,b,c \in G, (ab)c = a(bc)$.

METHOD OF CONTRUCTION

Let X be a non empty set and $W_X^{'}$ denote the free algebra over X, Now if multiplication is defined on the algebra $W_X^{'}$ as provided in [18] by

$$u \circ v = \alpha(u) + \alpha^2(v) \, \forall u, v \in W_X^{'}, \qquad (1)$$

Then, the set $W_X^{'}$ is an LA semigroup under the binary operation defined between $u \, and \, v$. The new model used in the above construction of Σ-Semigroup was given as follows:

$$u \circ v = \alpha(u)\alpha(v) + \alpha^2(uv) \;\; \forall \, u, v \in W_X^{'} \qquad (2)$$

and the new adjustment is made in order to make the construction smooth and accurate.

Recall that a groupoid must satisfy some conditions before it is consider as a left almost semigroup or right almost semigroup. In this work, we shall used these conditions to determine whether a particular groupoid is a left almost or right almost semigroup.

The two conditions are: $\forall \, a, b, c \in G, \, (a.b).c = (c.b).a \;\; and \;\; a.(b.c) = (c.b).a$ and are called left invertive and right invertive law respectively. In this work, left almost and right almost semi group will be denoted by LA and RA respectively.

Let us now test the properties as follows:

Example 1.1 Consider the set $F_5 = \{0, 1, 2, 3, 4\}$ under multiplication modulo 5. Define α by $\alpha(x) = 3x$, then we obtained the table below as LA Semigroup.

Table 1.1

.	0	1	2	3	4
0	0	4	3	2	1
1	3	2	1	0	4
2	1	0	4	3	2
3	4	3	2	1	0
4	2	1	0	4	3

From the above table, it can be seen that by referring back to the definition of multiplication of elements of $W_X^{'}$, $u \, and \, v$, the two statements are equivalent. Take for example the elements $4, 3 \, and \, 2$, thus, we have

$$\begin{aligned} &(4.3).2 = (2.3).4 \\ &\Rightarrow [(4).2] = [(3).4] \\ &\Rightarrow [(4.2)] = [(3.4)] \\ &\Rightarrow 0 = 0 \end{aligned}$$

One can also swap or take other elements from the set apart from 4, 3 and 2 just to confirm that the structure satisfies the law for all elements taken from it. For instance the following

$$\begin{aligned} &if \;\; a = 0, b = 1, c = 4, \; then \\ &(0.1)4 = (4.1)0 \\ &\Rightarrow 4.4 = 1.0 \\ &\Rightarrow 3 = 3 \end{aligned}$$

Or

$$\begin{aligned} &if \;\; a = 1, b = 2, c = 3, \; then \\ &(1.2)3 = (3.2)1 \\ &\Rightarrow 1.3 = 2.1 \\ &\Rightarrow 0 = 0 \end{aligned}$$

Now, the left invertive law holds in F_5 so F_5 is an LA semigroup, what now remains is to confirm whether the two statements given in the proposition above are true or not. Taking the first statement,

$(ab)c = b(ca)$, we see that choosing any three elements say *1, 4* and *2* in F_5 to represent $a, b \; and \; c$ respectively will give

$$(1.4).2 = (4.2).1$$
$$\Rightarrow [(4).2] = [(0).1]$$
$$\Rightarrow (4.2) = (0.1)$$
$$\Rightarrow 0 = 0$$

Similar result can be obtained from the second statement by just commuting $a \; and \; c$ in the left hand side of (i).

Example 3.7
Let W be an arbitrary set and defined $u \circ v = \alpha u \alpha v + \alpha^2 uv$ by changing the multiplication between the elements of the set we obtained the table below;

Table: 1.2

∘	0	1	2	3	4
0	0	0	0	0	0
1	0	3	1	4	2
2	0	1	2	3	4
3	0	4	3	2	1
4	0	2	4	1	3

It can be seen that by taking any three elements for example *3, 1,* and *2,* we have that

$$(3.1).4 = (4.1).3$$
$$\Rightarrow (4.4) = (2.3)$$
$$\Rightarrow 3 = 3$$

That is $(a.b).c = (c.a).b \; \forall a, b, c \in F_5$ which implies that the left invertive law holds in F_5. It can also be seen that

$$3.(1.4) = 4.(1.3)$$
$$\Rightarrow (3.2) = (4.4)$$
$$\Rightarrow 3 = 3$$

That is $(a.b).c = c.(b.a) \; \forall a, b, c \in F_5$ which implies that the right invertive law holds in F_5. So F_5 is both RA and LA semigroup. Hence the structure is a Σ-semigroup semigroup.

Corollary: a Σ-semigroup semigroup is medial.

Example 3.8 Let us now test for the medial law in F_5, thus,

$$(1.2)(3.4) = (1.3)(2.4)$$
$$\Rightarrow (1.1) = (4.4)$$
$$\Rightarrow 3 = 3$$

Also

$$(0.1)(4.3) = (0.4)(1.3)$$
$$\Rightarrow (0.4) = (0.4)$$
$$\Rightarrow 0 = 0$$

So the medial law holds in F_5, that is $(a.b)(c.d) = (a.c)(b.d) \; \forall a, b.c.d \in F_5$ the two statement given in proposition above can also be verified. That is

(i) $\quad (ab)c = b(ca)$

(ii) $\quad (ab)c = b(ac)$

The first statement gives

$$(3.1).4 = 1.(4.3)$$
$$\Rightarrow 4.4 = 1.1$$
$$\Rightarrow 3 = 3$$

While the second statement gives

$$(3.1).4 = 1.(3.4)$$
$$\Rightarrow 4.4 = 1.1$$
$$\Rightarrow 3 = 3$$

This shows that the two statements holds in the table, hence are equivalent in the table.

Corollary: A Σ-semigroup is paramedical. That is to say the statement

$$(ab)(cd) = (db)(ca) \; \forall \; a, b, c, d \in S$$

Example 3.8: consider a Σ-semigroup with 1dentity 8 as shown below

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Table: 1.3

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	8	7	6	5	4	3	2	1
2	0	7	5	3	1	8	6	4	2
3	0	6	3	0	6	3	0	6	3
4	0	5	1	6	2	7	3	8	4
5	0	4	8	3	7	2	6	1	5
6	0	3	6	0	3	6	0	3	6
7	0	2	4	6	8	1	3	5	7
8	0	1	2	3	4	5	6	7	8

From the above table, if $a = 1, b = 4, c = 8$, then

$$(1.4).8 = (8.4).1$$
$$\Rightarrow 5.8 = 4.1$$
$$\Rightarrow 5 = 5$$

Or
If $a = 8,\ b = 6, c = 4$, then

$$a = 8,\ b = 6, c = 4$$
$$(8.6).4 = (4.6).8$$
$$\Rightarrow 6.4 = 3.8$$
$$\Rightarrow 3 = 3$$

So this implies that the left invertive law holds. It can also be seen that the right invertive law holds, if $a = 1, b = 4, c = 8$, then
So this implies that the left invertive law holds. It can also be seen that the right invertive law holds, if $a = 1, b = 4, c = 8$, then

$$a = 8,\ b = 6, c = 4$$
$$8.(6.4) = 4.(6.8)$$
$$\Rightarrow 8.3 = 4.6$$
$$\Rightarrow 3 = 3$$

So the structure is a Σ-semigroup since both the right and the invertive law hold. We shall now test for the medial law, now if $a = 8,\ b = 6, c = 4, d = 5$, then
$$a = 8,\ b = 6, c = 4, d = 5$$
$$(8.6)(4.5) = (8.4)(6.5)$$
$$\Rightarrow 6.7 = 4.6$$
$$\Rightarrow 3 = 3$$

Or

$$a = 5,\ b = 2, c = 7, d = 3$$
$$(5.2)(7.3) = (5.7)(2.3)$$
$$\Rightarrow 8.6 = 1.3$$
$$\Rightarrow 6 = 6$$

So the medial law holds. It is also shown below that paramedical law holds in Σ-semigroup. Thus,

$$if\ \ a = 5,\ b = 2, c = 7, d = 3,\ then$$
$$(5.2)(7.3) = (3.2)(7.5)$$
$$\Rightarrow 8.6 = 3.1$$
$$\Rightarrow 6 = 6$$

So the medial law holds. It is also shown below that paramedical law holds in Σ-semigroup. Thus,

$$if\ \ a = 5,\ b = 2, c = 7, d = 3,\ then$$
$$(5.2)(7.3) = (3.2)(7.5)$$
$$\Rightarrow 8.6 = 3.1$$
$$\Rightarrow 6 = 6$$

Examples 1.4
The following tables are Σ-subSemigroup of S_8 and S_6 respectively Σ-subsemigroup of S

Table: 1.4 (a)

.	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0

Table: 1.4 (b)

.	0	1	2	3	4
0	0	0	0	0	0
1	0	2	4	0	2
2	0	4	2	4	0
3	0	0	0	0	0
4	0	2	4	0	2

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∑-semigroup as a graph structure

Consider a Σ-semigroup of order 5 i.e $S = \{0,1,2,3,4\}$, by defining a function f by $f(x) = x^2 \ \forall x \in S$, then the pairs of points for the relation are $\{(0,0),(1,4)(2,4),(3,0),(4,2)\}$, the adjacency matrix for the graph of the relation is

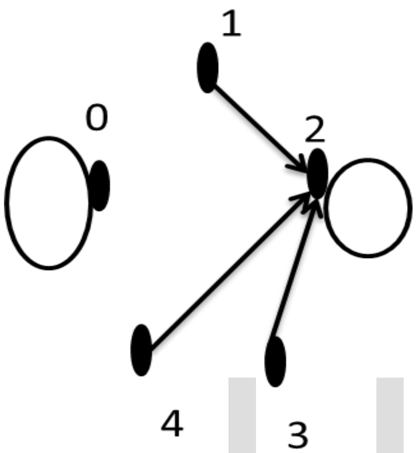

Fig: 1.1 Graph of $f(x) = x^2 \ \forall x \in S$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Consider another function defined in the set by $f(x) = x^3 \ \forall x \in S$ from which the following pairs of points were obtained $\{(0,0),(1,4),(2,2),(3,3),(4,1)\}$. The adjacency matrix and the graph are given below

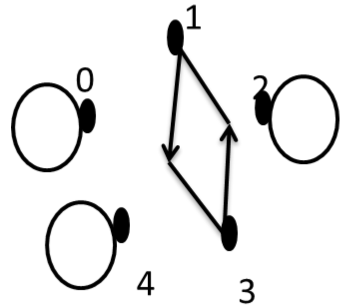

Figure: 1.2 Graph of $f(x) = x^3 \ \forall x \in S$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

If the function is defined as $f(x) = x^4 \ \forall x \in S$, then we have the following pairs of points, adjacenc matrix and graph $\{(0,0),(1,2),(2,2),(3,2),(4,2)\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

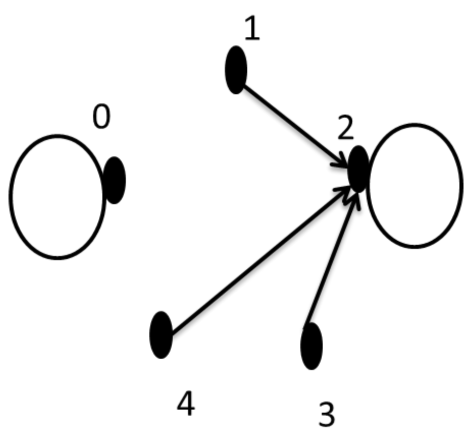

Fig: 1.3 Graph of $f(x) = x^4 \ \forall x \in S$

If $f(x)=x^5\ \forall x\in S$ then the following pairs of points adjacency matrix and graph are obtained $\{(0,0),(1,1),(2,2),(3,3),(4,4)\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

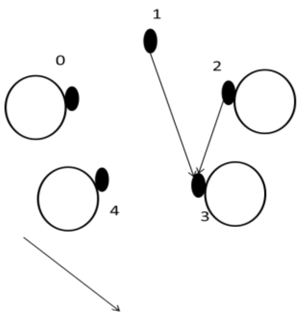

Figure 3.5 Graph of $f(x)=x^5\ \forall x\in S$

ng are pairs for the relation $\{(0,0),(1,3),(2,2),(3,2),(4,3)\}$. The adjacency matrix and the graph for the relation are

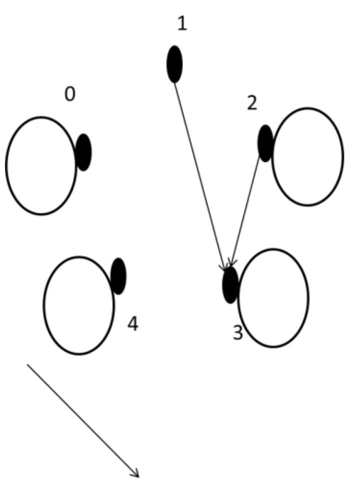

Fig: 1.5 Graph of $f(x)=x^5\ \forall x\in S$

$f(x)=x+x^2\ \forall x\in S$, then the following pairs of points were obtained from the relation $\{(0,0),(1,4),(2,4),(3,0),(4,2)\}$, the graph and the adjacency matrix are shown below.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

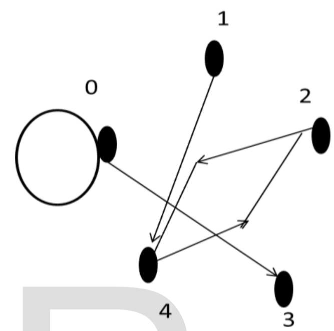

Figure 1.6 Graph of $f(x)=x+x^2\ \forall x\in S$,

If the function is defined as $f(x)=x+x^3\ \forall x\in S$, the following is obtained from the relation $\{(0,0),(1,0),(3,1),(4,0)\}$. The adjacency matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

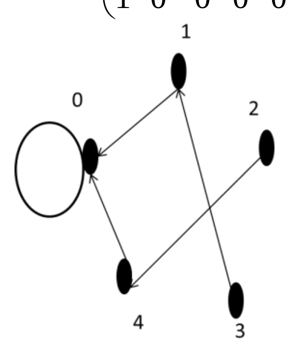

Figure 1.6 Graph of $f(x)=x+x^3\ \forall x\in S$

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If the function is defined to be $f(x)=x^2+x^4\ \forall x\in S$, the following pairs of points are obtained from the relation $\{(0,0),(1,0),(2,4),(3,4)(4,0)\}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

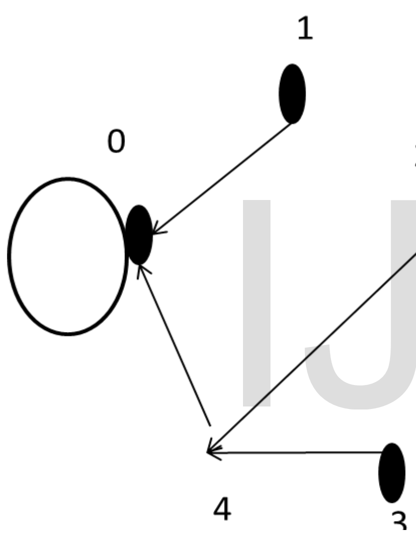

Figure 1.7 Graph of $f(x)=x^2+x^4\ \forall x\in S$

If the function is defined to be $f(x)=x^2+x^3\ \forall x\in S$ on a Σ-semigroup of order 9, then the following pairs are obtained from the $\{(0,0),(1,0),(2,4),(3,0),(4,3),(5,2),(6,0),(7,6),(8,7)\}$ the adjacency matrix and the graph for the relation are as shown in the figure below.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

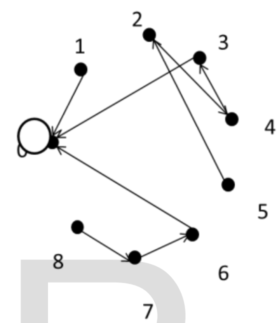

Figure 3.7 Graph of $f(x)=x^2+x^3\ \forall x\in S$

As it can be seen from the figure, the graph produces about three paths namely $0-1, 8760\ i.e\ 8-0$ and 52430 i.e $5-0$ path. It can be observed that the graph is connected and both the three paths came from different origins but they all have the same destination (0). It can be observed from the above that all the graphs the functions have directed edges hence are directed graphs and the raws of the adjacency matrix are linearly dependent vectors(by row operattion)

3.6 Connected and Complete graphs of a Σ-semigroup

A connected graph is a graph in which all the vertices are connected by the edges in one way or the other. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected b a unique edge [2]. Now we have the following lemma.

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Let $H = \{f(x) : x \in S\}$ (the set of images of elements of S under f)

Lemma: Let S be a Σ-semigroup and H be Σ-subsemigroup of S. the direct product SxH forms a connected graph.

Example 1.5 Consider Σ-semigroup of order 5 some graphs are shown on direct product of the Σ-semigroupand some of its subsets.

(i) For $f(x) = x^3 \; \forall x \in S,\; H = S$ and $G = S \times H = S \times S$ is a complete graph as it is shown in the figure below

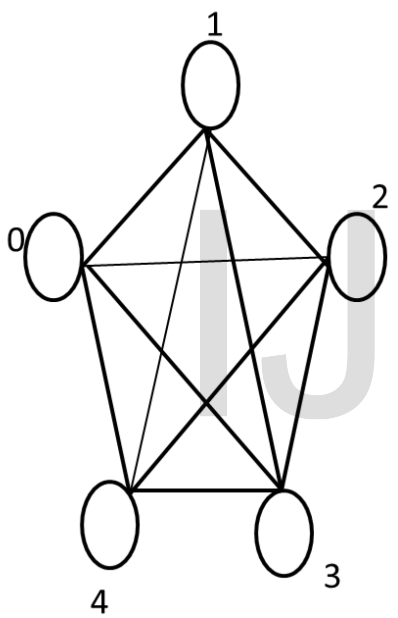

Figure 1.40 A 4-Simplex

$$A = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix} - \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix} =$$

$$(A-I)^2 = \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix} \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix} =$$

$$\begin{bmatrix} 4&3&3&3&3 \\ 3&4&3&3&3 \\ 3&3&4&3&3 \\ 3&3&3&4&3 \\ 3&3&3&3&4 \end{bmatrix}$$

$$\begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix}$$

Thus the graph is strongly connected and regular, where the values in the leading diagonal of the adjacency matrix show the number of edges incidented on each vertex of the graph.

$$\begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 \rightarrow \\ r_4 \\ r_5 \end{array} \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 0&1&-1&0&0 \\ 0&1&0&-1&0 \\ 0&1&0&0&-1 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 - r_2 \rightarrow \\ r_4 - r_2 \\ r_5 - r_2 \end{array}$$

$$\begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 0&1&-1&0&0 \\ 0&0&1&-1&0 \\ 0&0&1&0&-1 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 \\ r_4 - r_3 \\ r_5 - r_3 \end{array} \rightarrow \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 0&1&-1&0&0 \\ 0&0&1&-1&0 \\ 0&0&0&1&-1 \end{bmatrix} \begin{array}{l} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 - r_4 \end{array}$$

Thus, the matrix does not have zero row, hence the raws of the matrix are linearly independent vectors. In other words, the the raws of te matrix satisfies the set of equations

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= 0 \\ a_0 \quad + a_2 + a_4 &= 0 \\ a_1 - a_2 &= 0 \\ a_2 - a_3 &= 0 \\ a_3 - a_4 &= 0 \\ \Rightarrow a_1 = a_2 = a_3 = a_4 &= 0 \end{aligned}$$

Consider a case where $f(x) = x + x^3, \forall x \in S$ the graph G = $G = S \times H = S \times S$ for the relation is shown to be complete graph with the following properties.

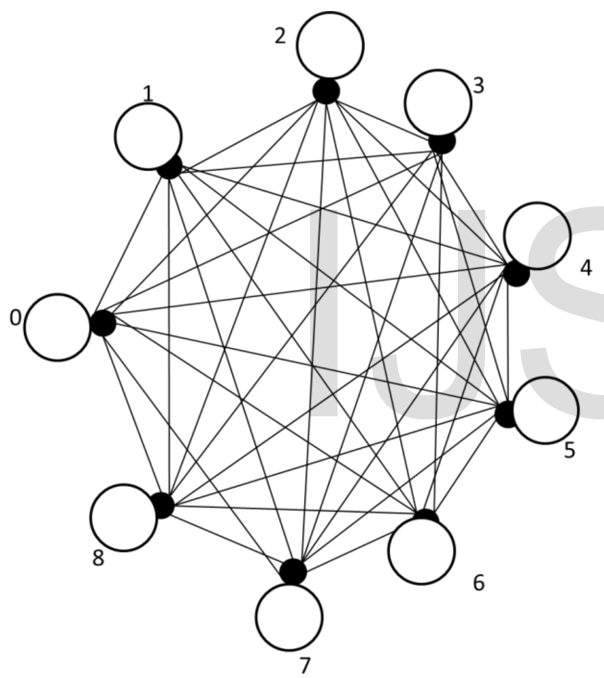

Figure 3.41 An 8-Simplex

Vertices = 9

Edges = $\dfrac{n(n-1)}{2} = 36$

Radius = 1(n is greater than 1)

Diameter = 1 (n is greater than 1)

Girth = 3 (n is greater than 2)

Chromatic number = 9

Chromatic index = 9 (n is odd)

The graph is 8-reular symmetric graph, it is vertex transitive, edge-transitive and strongly regular [25]. Such properties could also be obtained from figure 3.40 above.

According to [10], the degree of a vertex is the number of incident edges to that vertex. The distance between two vertices is the length of the shortest distance between them. The diameter of a graph is the maximum distance over all pairs of vertices. A graph with maximal degree Δ and diameter D is called a (Δ, D)-graph. The number of vertices in a graph is called order of the graph. So the degree of the above graph is Δ = 10 (loops are counted twice). One shall observe that all the vertices of the graph have the same degree which means that the graph is regular.

The eccentricity of a vertex v of a graph G is the maximum distance to every other vertex of the graph. That is: $e(s) = \max(\{dist(s,v) : v \in V\})$ (V is the set of vertices as usual

So diameter of a graph is the maximum eccentricity of any vertex in the graph i.e diameter is the length of the shortest path between the must distanced vertices in a graph. So to determine the diameter of the above graph, we first notice that the maximum distance between each pair of vertices is 1 so the greatest length of any of these paths is the diameter of the graph (the diameter of the graph is 1).

3.8 Adjacency and incidence matrices for the graph of a Σ-semigroup

Let *G* be a graph with *V(G)* = {1, . . . , *n*} and *E(G)* = {*e*1, . . . , *em*}. Suppose each edge of *G* is assigned an orientation, which is arbitrary but fixed. The (*vertex-edge*) incidence matrix of *G*, denoted by Q*(G)*, is the $n \times m$ matrix defined as follows. The rows and the columns of Q*G)* are indexed by *V(G)* and *E(G)*, respectively. The *(i, j)*-entry of Q*(G)* is 0 if vertex *i* and edge *e j* are not incident, and otherwise it is 1 or −1 according as *e j* originates or terminates at *i*,

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respectively. We often denote Q*(G)* simply by Q. Whenever we mention Q*(G)* it is assumed that the edges of *G* are oriented.
Consider the graph for a function defined by $f(x)=x^2+2\ \forall x \in S$. the incidence matrix for the graph is given and is denoted by Q

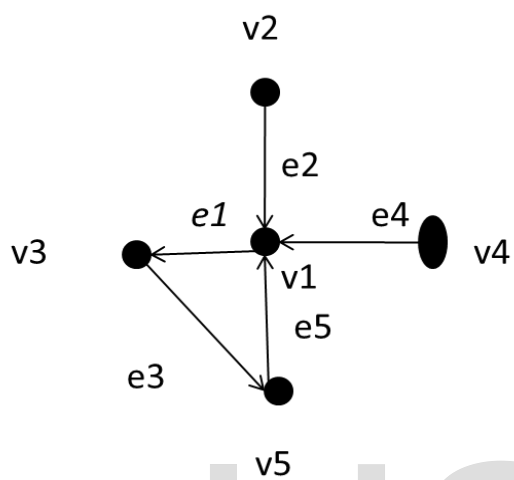


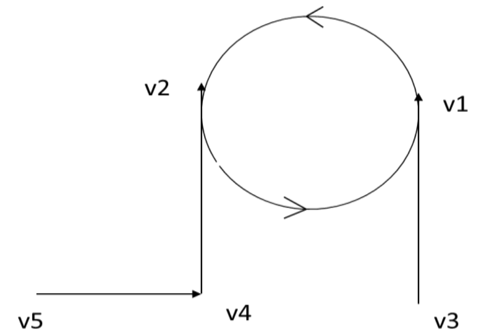
Figure 3.42

$$Q=\begin{pmatrix} -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$$

It can be observed from Q above that the column sums are zero and hence the rows of Q form linearly dependent vectors

A matrix is said to be totally unimodular if the determinant of any square submatrix of the matrix is either 0 or ± 1. So it follows that the matrix Q above is totally unimodular.

Consider another graph constructed from a function defined by $f(x)=x^2+x+1\ \forall x \in S$



$$Q=\begin{pmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Fig: 1.43

So the column sums of Q are zero hence rows of Q are linearly dependent vectors this assures that for all the cases graph may be, the raws of the incident matrix of the graph form linearly dependent vectors because the column sum for each column of the matrix is always zero.

The adjacency matrix A of the above graph is also a matrix having all its entries as1s since there is an edge between any pair of vertices. A matrix M for the graph is also obtained as shown below.

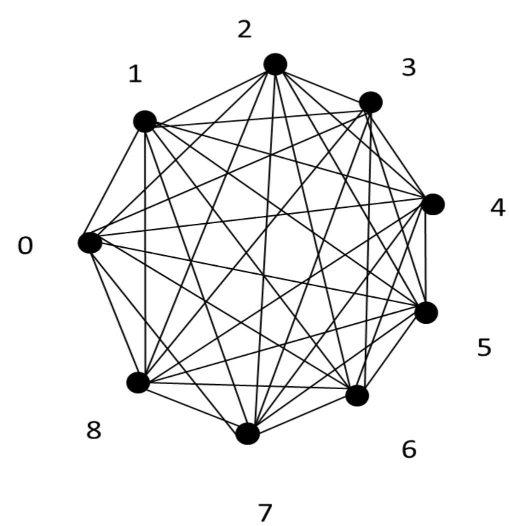


Fig: 1.44

On multiplying the matrix $A-I=M$ by itself up to $n-1$times, for the first time one will find that all the zero entries have taken value greater than zero which is showing that the graph is strongly connected and the values are showing number of edges that incident on each vertex. Thus,

$$(A-I)^2=\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\1&1&0&1&1&1&1&1&1\\1&1&1&0&1&1&1&1&1\\1&1&1&1&0&1&1&1&1\\1&1&1&1&1&0&1&1&1\\1&1&1&1&1&1&0&1&1\\1&1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&1&0\end{pmatrix}\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\1&1&0&1&1&1&1&1&1\\1&1&1&0&1&1&1&1&1\\1&1&1&1&0&1&1&1&1\\1&1&1&1&1&0&1&1&1\\1&1&1&1&1&1&0&1&1\\1&1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&1&0\end{pmatrix}=\begin{pmatrix}8&7&7&7&7&7&7&7&7\\7&8&7&7&7&7&7&7&7\\7&7&8&7&7&7&7&7&7\\7&7&7&8&7&7&7&7&7\\7&7&7&7&8&7&7&7&7\\7&7&7&7&7&8&7&7&7\\7&7&7&7&7&7&8&7&7\\7&7&7&7&7&7&7&8&7\\7&7&7&7&7&7&7&7&8\end{pmatrix}$$

Let us now observe the properties of rows of the matrix by applying row operations on the matrix to determine whether the matrix has zero rows or not.

$$A-I=\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\1&1&0&1&1&1&1&1&1\\1&1&1&0&1&1&1&1&1\\1&1&1&1&0&1&1&1&1\\1&1&1&1&1&0&1&1&1\\1&1&1&1&1&1&0&1&1\\1&1&1&1&1&1&1&0&1\\1&1&1&1&1&1&1&1&0\end{pmatrix}\rightarrow\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&1&0&-1&0&0&0&0&0\\0&1&0&0&-1&0&0&0&0\\0&1&0&0&0&-1&0&0&0\\0&1&0&0&0&0&-1&0&0\\0&1&0&0&0&0&0&-1&0\\0&1&0&0&0&0&0&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3-r_2\\\rightarrow r_4-r_2\\\rightarrow r_5-r_2\\\rightarrow r_6-r_2\\\rightarrow r_7-r_2\\\rightarrow r_8-r_2\\\rightarrow r_9-r_2\end{array}$$

$$?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&1&0&-1&0&0&0&0\\0&0&1&0&0&-1&0&0&0\\0&0&1&0&0&0&-1&0&0\\0&0&1&0&0&0&0&-1&0\\0&0&1&0&0&0&0&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4-r_3\\\rightarrow r_5-r_3\\\rightarrow r_6-r_3\\\rightarrow r_7-r_3\\\rightarrow r_8-r_3\\\rightarrow r_9-r_3\end{array}\quad ?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&0&1&-1&0&0&0&0\\0&0&0&1&0&-1&0&0&0\\0&0&0&1&0&0&-1&0&0\\0&0&0&1&0&0&0&-1&0\\0&0&0&1&0&0&0&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4\\\rightarrow r_5-r_4\\\rightarrow r_6-r_4\\\rightarrow r_7-r_4\\\rightarrow r_8-r_4\\\rightarrow r_9-r_4\end{array}$$

$$?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&0&1&-1&0&0&0&0\\0&0&0&0&1&-1&0&0&0\\0&0&0&0&1&0&-1&0&0\\0&0&0&0&1&0&0&-1&0\\0&0&0&0&1&0&0&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4\\\rightarrow r_5\\\rightarrow r_6-r_5\\\rightarrow r_7-r_5\\\rightarrow r_8-r_5\\\rightarrow r_9-r_5\end{array}\quad ?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&0&1&-1&0&0&0&0\\0&0&0&0&1&-1&0&0&0\\0&0&0&0&0&1&-1&0&0\\0&0&0&0&0&1&0&-1&0\\0&0&0&0&0&1&0&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4\\\rightarrow r_5\\\rightarrow r_6\\\rightarrow r_7-\\\rightarrow r_8-\\\rightarrow r_9-\end{array}$$

$$?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&0&1&-1&0&0&0&0\\0&0&0&0&1&-1&0&0&0\\0&0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&1&-1&0\\0&0&0&0&0&0&1&0&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4\\\rightarrow r_5\\\rightarrow r_6\\\rightarrow r_7\\\rightarrow r_8-r_7\\\rightarrow r_9-r_7\end{array}\quad ?\begin{pmatrix}0&1&1&1&1&1&1&1&1\\1&0&1&1&1&1&1&1&1\\0&1&-1&0&0&0&0&0&0\\0&0&1&-1&0&0&0&0&0\\0&0&0&1&-1&0&0&0&0\\0&0&0&0&1&-1&0&0&0\\0&0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&1&-1&0\\0&0&0&0&0&0&0&1&-1\end{pmatrix}\begin{array}{l}\rightarrow r_1\\\rightarrow r_2\\\rightarrow r_3\\\rightarrow r_4\\\rightarrow r_5\\\rightarrow r_6\\\rightarrow r_7\\\rightarrow r_8\\\rightarrow r_9-\end{array}$$

Since the matrix has non zero rows, then all the rows are linearly independent vectors. In other words, the rows of the matrix satisfies the following set of equations.

$$\begin{aligned}
a_1+a_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_9 &= 0\\
a_0 \quad +a_2+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+a_9 &= 0\\
a_1-a_2 &= 0\\
a_2-a_3 &= 0\\
a_3-a_4 &= 0\\
a_4-a_5 &= 0\\
a_5-a_6 &= 0\\
a_6-a_7 &= 0\\
a_7-a_8 &= 0\\
a_8-a_9 &= 0
\end{aligned}$$

$$\Rightarrow a_0=a_1=a_2=a_3=a_4=a_5=a_6=a_7=a_9=0$$

So the rows of the adjacency matrix are linearly independent vectors.

Conclusion

It is clear that the new structure of a locally associated algebraic structure (∑-semigroup) possess the properties of left almost semigroup and right almost semigroup respectively. By defining a function on the new structure of ∑-semigroup the results established the formation of adjacency matrix which helped to show that the rows of the ∑-semigroup are linearly dependent.

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